J Stat Phys (2007) 129: 809–832
DOI 10.1007/s10955-007-9405-y

# Superbosonization Formula and its Application to Random Matrix Theory

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Received: 21 May 2007 / Accepted: 12 August 2007 / Published online: 15 September 2007


**Abstract** Starting from Gaussian random matrix models we derive a new supermatrix field theory model. In contrast to the conventional non-linear sigma models, the new model is applicable for any range of correlations of the elements of the random matrices. We clarify the domain of integration for the supermatrices, and give a demonstration of how the model works by calculating the density of states for an ensemble of almost diagonal matrices. It is also shown how one can reduce the supermatrix model to the conventional sigma model.

## 1 Introduction

The non-linear supersymmetric sigma model (NL$\sigma$M) [1] has been very fruitful in describing quantum systems with quenched disorder [2]. Derived originally [2] from a model of noninteracting electrons in a Gaussian random potential, NL$\sigma$M was later constructed in its zero-dimensional version directly [3] from the Wigner–Dyson ensembles [4] of random matrix theory. This gave the first demonstration of equivalence between these two problems in the low-frequency domain. Furthermore, even for problems where the conventional Wigner–Dyson random matrix theory is sufficient (see, e.g. [4–6]) NL$\sigma$M turned out to be a competitive method of calculation compared to the standard approach based on orthogonal

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polynomials [4]. Let us mention parametric level statistics [7] as a beautiful example of new results which were obtained in the field of classical RMT using NL$\sigma$M.

The most significant applications of NL$\sigma$M are, however, in the realm beyond classical RMT, such as corrections to the Wigner–Dyson theory in systems with diffusive dynamics [8–10], and random matrix ensembles with a probability distribution not invariant under the full group of unitary transformations. The domain of application of such random matrix ensembles is rather wide. For example, band random matrices are equivalent to thick disordered wires [2, 11], where strong localization was first established using NL$\sigma$M. Another example is the Critical Power-Law Band Random Matrix (CPLBRM) ensemble [12]. An important property of eigenvector statistics in this ensemble is multifractality [12–14], which makes CPLBRM a good model to qualitatively (and in many cases quantitatively) describe the critical states near the Anderson transition [13, 15]. In the analysis of all these models NL$\sigma$M was an extremely useful guide, although limited, as NL$\sigma$M is not always exactly equivalent to the model being studied. The reason for this limitation is the saddle-point approximation which is used in deriving NL$\sigma$M. In the case of band random matrices the band width $B$ has to be large in order for the saddle-point approximation to be valid. For CPLBRM this implies weak multifractality, while for band random matrices it means that the localization length is much larger than the lattice scale. In both cases localization effects are in a certain sense weak.

There are several important problems which lie beyond the applicability of NL$\sigma$M. These include the properties of one-dimensional disordered chains and the problem of critical eigenfunctions and level statistics in systems with strong multifractality. The Hamiltonian of the first example is just a tridiagonal random matrix, while a representative model for the second example is CPLBRM with a small bandwidth $B \sim 1$. Both of these are examples of a class of Gaussian ensembles of *almost diagonal random matrices* which were recently defined and studied in Refs. [16–19]. In order to attack these problems by the supersymmetry method, one needs an exact representation of the corresponding correlation functions in terms of a supermatrix field theory.

Such a representation called 'superbosonization' has recently been suggested in Refs. [20, 21]. Similar ideas had been put forward earlier by Lehmann, Saher, Sokolov, and Sommers [22], and by Hackenbroich and Weidenmüller [23]. Here we give a decent derivation of the key formula of superbosonization, including such aspects as integration measure and domain of integration. We also work out a simple application of the formula involving Gaussian random matrices.

Given any function $f(\psi \otimes \bar{\psi})$ of the tensor product of a supervector $\psi$ and its conjugate $\bar{\psi}$, the superbosonization formula essentially converts the integral of $f$ over $\psi$ to an integral over a supermatrix $Q$ akin to $\psi \otimes \bar{\psi}$. Thus it provides a direct way to rewrite in terms of $Q$ the so-called $\psi$-functional which arises after ensemble averaging the initial free-field representation of disordered one-particle systems. We wish to stress that the conversion from $\psi$ to $Q$ is *exact*. It is also non-trivial, as the number of independent variables in the supermatrix $Q$ may be smaller or larger than the number of independent variables in the initial supervector representation. We note that a workable [19, 24] integration formula which converts integrals over $\psi$ into integrals over some supermatrix $\tilde{Q}$ (different from $Q$) can be constructed by a mere change of variables [24] without changing the number of variables. What we describe in this paper is not simply a change of variables.

For ease of discussion, we consider disordered one-particle Hamiltonians in the form of a random Hermitian matrix $H$ with independent, Gaussian-distributed entries $H_{ij}$ $(i, j = 1, \ldots, N)$ characterized by a certain variance matrix $C_{ij} = c(|i - j|) = \langle |H_{ij}|^2 \rangle$. The special case of classical Wigner–Dyson random matrix theory is obtained by setting $c(|i - j|) =$

const. While this has the largest possible symmetry group, $\mathrm{U}(N)$, our ensembles with non-constant $c(|i-j|)$ generically have symmetry group $\mathrm{U}(1)^N$. From the perspective of possible applications, the main outcome of this paper is an exact field-theoretical representation of correlation functions for the generic case of $\mathrm{U}(1)^N$ symmetry.

To exert better control on the mathematics, and also with a view to certain applications, we consider random matrix models with a local gauge symmetry $\mathrm{U}(n)^{N/n}$. The random matrix Hamiltonians with matrix elements $H_{ij}^{ab}$ in such models have a block structure with $i, j = 1, \ldots, M := N/n$ and $a, b = 1, \ldots, n$. The probability distribution function is of the form

$$P(H) = \exp\left( -\frac{1}{2} \sum_{i,j=1}^{M} c(|i-j|)^{-1} \sum_{a,b=1}^{n} H_{ij}^{ab} H_{ji}^{ba} \right).$$

Setting $M = 1$, $n = N$ gives the classical Gaussian Unitary Ensemble (GUE) of Wigner and Dyson, while the case $n = 1$, $M = N$ corresponds to a generic Gaussian random matrix model.

It turns out that the precise form of the superbosonization formula depends in a crucial way on how $n$ compares with the number of commuting components $q$ in the supervector $\psi$. The formula agrees with the naive expectation [20, 21] if $q \leq n$ (as is the case, e.g., for the *mean* density of states (DoS) of a generic Gaussian Hermitian random matrix ensemble, where $q = n = 1$) but has a different form if $q > n$ (which is of relevance to the DoS *two-point* correlation function of a generic Gaussian Hermitian random matrix ensemble, where $q = 2$, $n = 1$). Establishing and interpreting this peculiar fact is the main message of this paper.

The structure of the paper is as follows. In Sect. 2 we derive the superbosonization formula and the corresponding supermatrix field theory. Our detailed treatment makes it clear that the formula changes as one goes from one-point functions (i.e., the mean density of states, DoS), to two-point functions (e.g., the DoS-DoS correlation function), to three-point functions, and so on. In Sect. 3 we use the formalism to compute corrections to the DoS of the Gaussian ensemble of almost diagonal random matrices. These were obtained earlier in Ref. [18] by another method. In Sect. 4 we show how the new supermatrix field theory can be reduced to the conventional NL$\sigma$M. In Sect. 5 we give a summary of the main results and outline some open problems. In particular we discuss the possibility of a unified description of the cases $q > n$ and $q \leq n$.

Although the present paper illustrates the use of the superbosonization method only at the very simple example of the density of states, in the parameter range $q \leq n$ there exist no serious obstacles for the application of the method to more challenging problems such as level correlation functions and eigenfunction statistics. Some applications of this kind are currently being worked out and will be presented in forthcoming articles.

## 2 The Superbosonization Formula

The first few steps of the method presented here are similar to previous derivations of the non-linear sigma model for random matrices [2, 3]. However, while the usual scheme employs a Hubbard–Stratonovich transformation after the initial step of averaging over the random matrix, here we will follow a different route avoiding the Hubbard–Stratonovich transformation.

### 2.1 Review of Basic Steps

We begin with a generating function $Z_0[J]$,

$$Z_0[J] = \int D\bar{\psi} D\psi \exp\left(-\mathrm{i} \sum\nolimits_{i,j} \bar{\psi}_i s \mathcal{H}^J_{ij} \psi_j\right), \qquad \mathcal{H}^J_{ij} = H_{ij} - \delta_{ij}(\mathcal{E} + J_i), \tag{1}$$

where the $\psi_i$ are supervectors (with components arranged as column vectors), $\bar{\psi}_i$ are the conjugate row supervectors, $H_{ij}$ are the matrix elements of our random matrix $H$, $J_i$ is the source field, and $\mathcal{E}$ is a diagonal matrix containing the energy parameters of the problem. The diagonal matrix $s$ with entries $\pm 1$ keeps track of whether the Green's functions to be computed from the generating function (1) are retarded (i.e., have an energy parameter with positive imaginary part) or advanced (negative imaginary part). A scalar product or sum over components $\bar{\psi}_i s \psi_j \equiv \sum_\lambda \bar{\psi}^\lambda_i s^\lambda \psi^\lambda_j$ in the exponential is understood. The number of components of the supervectors $\psi_i$ is chosen according to the physical quantity under consideration. To compute the ensemble average of a product of $p$ Green's functions or matrix entries of the resolvent operator $(E - H)^{-1}$, one uses supervectors with $p$ anti-commuting and $p$ commuting components. In particular, to calculate the two-level correlation function one chooses $p = 2$. In the case of the density of states it suffices to use $p = 1$. In the most general situation (going beyond the calculation of Green's functions or resolvents) each $\psi_i$ has $p$ anti-commuting and $q$ commuting components.

In order to make the presentation self-contained, we now give more details concerning the supervectors $\psi_i$ and the matrix $s$. The definitions are the same for every site, so we temporarily drop the site index $i$ from our notation. We then have $\psi = \binom{\chi}{S}$ and $\bar{\psi} = (\bar{\chi}\ \ \bar{S})$ where

$$\chi = \begin{pmatrix} \chi^1 \\ \vdots \\ \chi^p \end{pmatrix}, \qquad S = \begin{pmatrix} S^1 \\ \vdots \\ S^q \end{pmatrix}, \qquad \bar{\chi} = (\chi^{1*} \quad \dots \quad \chi^{p*}), \qquad \bar{S} = (S^{1*} \quad \dots \quad S^{q*}),$$

and $\chi^\mu$ and $S^\nu$ are anti-commuting and commuting variables, respectively. $S^{\nu *}$ is the complex conjugate of the complex variable $S^\nu$. The integral (1) is carried out w.r.t. the Berezin superintegral form

$$D\bar{\psi} D\psi \equiv (2\pi)^{-p}\, d\bar{S} dS\, \partial_{\bar{\chi}} \partial_\chi,$$

$$d\bar{S} dS = \prod_{\nu=1}^{q} 2\, d\mathfrak{Re}(S^\nu)\, d\mathfrak{Im}(S^\nu), \qquad \partial_{\bar{\chi}} \partial_\chi = \prod_{\mu=1}^{p} \frac{\partial^2}{\partial \chi^{\mu *} \partial \chi^\mu},$$

with the domain of integration for each commuting variable $S^\nu$ being the complex numbers, $\mathbb{C}$. The integration measure for the multi-site problem (1) is of course $D\bar{\psi} D\psi = \prod_i D\bar{\psi}_i D\psi_i$ (index $i$ reinstated). To specify the sign matrix $s$, let $q = q_+ + q_-$ and assume that the first $q_+$ (last $q_-$) energy parameters multiplying the commuting field components $S^\nu$ in (1) have a positive (resp. negative) real part. Then we choose $s$ to be the diagonal matrix

$$s = \mathrm{diag}(\mathbf{1}_p, \mathbf{1}_{q_+}, -\mathbf{1}_{q_-}).$$

(Note that there is absolutely no gain from introducing unnecessary minus signs in the fermion-fermion sector.)

The next step is to take the average over the random matrix using the Gaussian distribution

$$\langle\cdots\rangle_H \equiv \mathcal{N}\int dH(\cdots)\exp\left(-\frac{1}{2}\sum\nolimits_{i,j}|H_{ij}|^2/C_{ij}\right),$$

where the coefficients $C_{ij}$ are real, non-negative, and symmetric under exchanging $i \leftrightarrow j$, and $\mathcal{N}$ is the normalization constant ensuring $\langle 1\rangle_H = 1$. Of course $H_{ij}^* = H_{ji}$ by Hermiticity. The integration measure is

$$dH := \prod_k dH_{kk}\prod_{i<j} 2\,d\mathfrak{Re}(H_{ij})\,d\mathfrak{Im}(H_{ij}),$$

where we are using a normalization convention which will be in force throughout the paper.

An equivalent way of characterizing such a Gaussian distribution is by means of its Fourier transform with respect to some arbitrary commuting parameters $K_{ij}$:

$$\left\langle\exp\left(-\mathrm{i}\sum_{i,j}H_{ij}K_{ji}\right)\right\rangle_H = \exp\left(-\frac{1}{2}\sum_{i,j}C_{ij}K_{ij}K_{ji}\right). \tag{2}$$

The supervector integral representation (1) of the generating function $Z_0[J]$ allows us to average immediately over the random matrix $H$. Using (2) for $K_{ji} \equiv \bar\psi_i s\psi_j$, the average $Z[J]$ of the generating function is

$$\begin{aligned} Z[J] &\equiv \langle Z_0[J]\rangle_H \\ &= \int D\bar\psi D\psi\exp\left(\mathrm{i}\sum\nolimits_i \bar\psi_i s(\mathcal{E}+J_i)\psi_i - \frac{1}{2}\sum\nolimits_{i,j}C_{ij}(\bar\psi_i s\psi_j)(\bar\psi_j s\psi_i)\right). \end{aligned} \tag{3}$$

Up to now, everything was more or less standard. In the next step, however, we shall depart from the standard method [2], where one proceeds by making a Hubbard–Stratonovich transformation. Instead, we will carry out a certain change of integration variables. The change-of-variables formula to be introduced below bears some resemblance to the well-known Weyl integration formula, by which one reduces the integral over a conjugation-invariant function of a matrix to an integral over the eigenvalues of the matrix.

From the property of cyclic invariance of the supertrace, one has the identities

$$\mathrm{STr}(\psi_i\otimes\bar\psi_i)s = \bar\psi_i s\psi_i, \qquad \mathrm{STr}(\psi_i\otimes\bar\psi_i)s(\psi_j\otimes\bar\psi_j)s = (\bar\psi_i s\psi_j)(\bar\psi_j s\psi_i).$$

Reading these identities backwards, one reorganizes the exponential of the integrand in (3) as

$$\mathrm{i}\sum\nolimits_i \mathrm{STr}\,(\psi_i\otimes\bar\psi_i)s\,(\mathcal{E}+J_i) - \frac{1}{2}\sum\nolimits_{i,j}C_{ij}\,\mathrm{STr}(\psi_i\otimes\bar\psi_i)s(\psi_j\otimes\bar\psi_j)s,$$

which prompts a change of integration variables

$$\psi_i\otimes\bar\psi_i \to Q_i, \tag{4}$$

where $Q_i$ is a supermatrix. Note that while the sign matrix $s$ does appear in the integrand, it will have *no* influence on the definition of the integration variable $Q_i$. Thus the domain of integration for $Q_i$ will be independent of the types of Green's functions (retarded or advanced) which are being generated. This is a major difference from the traditional method using the Hubbard–Stratonovich transformation.

### 2.2 Heuristic Approach

The change of variables $\psi_i \otimes \bar{\psi}_i \to Q_i$ works in the same way at every site $i$. Let us therefore focus on a fixed site and simplify the notation by dropping the index $i$ for now.

In past work on the subject the desired change of variables (4) was brought about by an unprecedented and unexplained method, in the context of supermatrices, involving the $\delta$-function. The logic went something like this: to transform a Berezin superintegral such as $\int D\bar{\psi} D\psi f(\psi \otimes \bar{\psi})$, you insert

$$1 \stackrel{?}{=} \int DQ\delta(Q - \psi \otimes \bar{\psi})$$

under the integral sign. Here $Q$ is a supermatrix with an unspecified domain of integration. Then you reverse the order of integration of $Q$ and $\psi$ to write

$$\int D\bar{\psi} D\psi f(\psi \otimes \bar{\psi}) = \int DQ J(Q) f(Q),$$

$$J(Q) \stackrel{?}{=} \int D\bar{\psi} D\psi \delta(Q - \psi \otimes \bar{\psi}) \stackrel{?}{=} \mathrm{SDet}(Q).$$

This kind of manipulation with the $\delta$-function requires justification, which has never been provided. To give a simple indication of what the issue is, consider the fermion-fermion sector for the case of the two-level correlation function, i.e., put $p = 2$ and $q = 0$, in which case we are dealing with $2 \times 2$ matrices

$$\chi \otimes \bar{\chi} = \begin{pmatrix} \chi^1\chi^{1*} & \chi^1\chi^{2*} \\ \chi^2\chi^{1*} & \chi^2\chi^{2*} \end{pmatrix}, \qquad Q = \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}.$$

For concreteness let us say that $f(\psi \otimes \bar{\psi}) = f(\chi \otimes \bar{\chi}) = \chi^1\chi^{1*}\chi^2\chi^{2*}$. Now, in attempting to decide what function $f(Q)$ is to be placed in the integrand we encounter an ambiguity. If we group the anti-commuting variables as $f = (\chi^1\chi^{1*})(\chi^2\chi^{2*})$ we may be inclined to choose $f(Q) = Q^{11}Q^{22}$, but it is equally valid to reorder the anti-commuting variables as $f = -(\chi^1\chi^{2*})(\chi^2\chi^{1*})$, which would suggest $f(Q) = -Q^{12}Q^{21}$. Perhaps a linear combination $f(Q) = Q^{11}Q^{22} - t(Q^{11}Q^{22} + Q^{12}Q^{21})$ (with $t \in \mathbb{C}$) is the good choice to make. All of these do the necessary job of returning the given function $f = \chi^1\chi^{1*}\chi^2\chi^{2*}$ on making the substitution $Q \to \chi \otimes \bar{\chi}$. Thus, unless the matrix $Q$ is constrained by $Q^{11}Q^{22} + Q^{12}Q^{21} \equiv 0$, an arbitrary parameter $t$ enters into the calculation.

At this stage it must be recalled that in mathematics there exists no such thing as the *$\delta$-function*. What does exist and can be made sense of is the *$\delta$-distribution*, say $\delta_x$, which is the linear functional that evaluates test functions at the point $x$:

$$f \mapsto \delta_x[f] \equiv \int \delta(x - y) f(y)\, dy := f(x).$$

In our example, however, we are dealing with the element $f = \chi^1\chi^{1*}\chi^2\chi^{2*}$ of a Grassmann algebra—not with a function defined on points. It is unclear how we can evaluate a $\delta$-distribution over these variables. In other words, the meaning behind

$$\delta(Q - \chi \otimes \bar{\chi}) \stackrel{?}{=} \delta(Q^{11} - \chi^1\chi^{1*})\delta(Q^{22} - \chi^2\chi^{2*})\delta(Q^{12} - \chi^1\chi^{2*})\delta(Q^{21} - \chi^2\chi^{1*}),$$

viewed as a 'function' of the Grassmann variables, is undefined. No meaning exists in current mathematics.

In the simple case under consideration, it is not difficult to work out everything by hand and achieve control of the situation. Let us take the integration domain for the $2 \times 2$ matrix $Q$ to be the unitary group $\mathrm{U}(2)$ – or any other 4-dimensional closed submanifold of $\mathrm{GL}_2(\mathbb{C})$ in the same homology class as $\mathrm{U}(2)$. Then if $d\,\mathrm{vol}_2(Q)$ denotes a Haar measure on $\mathrm{U}(2)$, it can be shown that

$$\int_{\mathrm{U}(2)} d\,\mathrm{vol}_2(Q) J(Q)(Q^{11}Q^{22} + Q^{12}Q^{21}) = 0, \quad J(Q) = \mathrm{Det}^{-1}(Q).$$

So, although bosonization of $f = \chi^1\chi^{1*}\chi^2\chi^{2*}$ is an ambiguous procedure leading to any one of the one-parameter family of functions $F_t(Q) = Q^{11}Q^{22} - t(Q^{11}Q^{22} + Q^{12}Q^{21})$, the ambiguity, at least in this case, disappears at the level of integration provided we use $J(Q) = \mathrm{Det}^{-1}(Q)$ and the integration domain $\mathrm{U}(2)$.

This does not solve the problem of what to do when the Grassmann variables are more numerous or when considering the full supersymmetric situation which is rendered more complicated by the presence of commuting variables. It is clear that the same ambiguity occurs in the general case: for any given function $f(\psi \otimes \bar{\psi})$, there exist many choices of function $F(Q)$ such that $F(Q)|_{Q \to \psi \otimes \bar{\psi}}$ is equal to $f(\psi \otimes \bar{\psi})$. Given all these possible functions $F(Q)$ with which the superbosonization formula

$$\int D\bar{\psi} D\psi f(\psi \otimes \bar{\psi}) = \int DQ J(Q) F(Q)$$

appears to be true, we may think we can use any one of them, or we may attempt to impose some constraint on $Q$. Fortunately, recent mathematical work [25] has given a complete solution to this problem for a restricted range of parameter values. In the next subsection, we will present a summary of these results. Afterwards, will give a detailed proof of the superbosonization formula for a special but important case.

### 2.3 Rigorous Result

In order to control the mathematics, let us assume that there is a block or granular structure in the variances $C_{ij}$. By this we mean if $i$ is a multi-index $i = (I, a)$ where $a = 1, \ldots, n$, then

$$C_{ij} = C_{I,a;I',a'} = c_{II'}\delta_{aa'},$$

independent of $a, a'$. If so, then after grouping the terms appropriately, the integrand depends only on the sums $\sum_{a=1}^n \psi_{I,a} \otimes \bar{\psi}_{I',a}$. In this way the integer $n$ (sometimes referred to as the number of orbitals) is introduced as an additional parameter of our problem. Results stated in this subsection are valid only in the range $n \geq q$. Later, we will suggest how to recover the important case of $n = 1$ when $q > 1$.

To keep the notation simple, we return to using a single index $i$ as a label for our supervectors $\psi_i$. In keeping with the discussion above, we consider functions $f$ of the sum $\sum_{i=1}^n \psi_i \otimes \bar{\psi}_i$. Our goal is to transform (by superbosonization) the integral of such a function,

$$\int f \equiv \int D\bar{\psi} D\psi f\left(\sum_{i=1}^n \psi_i \otimes \bar{\psi}_i\right), \quad D\bar{\psi} D\psi \equiv \prod_{i=1}^n D\bar{\psi}_i D\psi_i. \tag{5}$$

Let us address the general case of supervectors $\psi_i = \binom{\chi_i}{S_i}$ that have $p$ anti-commuting and $q$ commuting components (for $\chi_i$ and $S_i$ respectively). Then $\sum_{i=1}^n \psi_i \otimes \bar{\psi}_i$ corresponds to a supermatrix

$$Q = \begin{pmatrix} A & \sigma \\ \tau & B \end{pmatrix},$$

where the blocks $A$ and $B$ are square matrices of size $p \times p$ and $q \times q$ with commuting variables as entries; while $\sigma$ and $\tau$ are rectangular matrices of size $p \times q$ and $q \times p$ respectively and these have anti-commuting entries.

Let $F$ now be any function of the supermatrix $Q$ so that on making the substitution $Q \to \sum \psi_i \otimes \bar{\psi}_i$ the function $F(Q)$ becomes equal to the given function $f(\sum \psi_i \otimes \bar{\psi}_i)$. We also assume that $n \geq q$. Then we claim that the following equality of integrals holds:

$$\int D\bar{\psi}\, D\psi f\left(\sum_{i=1}^n \psi_i \otimes \bar{\psi}_i\right) = \frac{\operatorname{vol} \mathrm{U}(n)}{\operatorname{vol} \mathrm{U}(n+p-q)} \int_D DQ\, \mathrm{SDet}^n(Q) F(Q), \tag{6}$$

provided $f$ decreases sufficiently fast at infinity so that the integral on the left-hand side exists.

We now define all terms on the right-hand side of the identity (6). The domain of integration $D$ is the unitary group $\mathrm{U}(p)$ for $A$ and is the positive Hermitian $q \times q$ matrices, $\mathrm{Herm}^+(q)$, for $B$. Thus $D = \mathrm{U}(p) \times \mathrm{Herm}^+(q)$. The Berezin superintegral form $DQ$ is given by

$$DQ := (2\pi)^{-pq}\, d\operatorname{vol}_p(A)\, d\operatorname{vol}_q(B) \partial_\tau \partial_\sigma \operatorname{Det}^q(A - \sigma B^{-1}\tau) \operatorname{Det}^p(B - \tau A^{-1}\sigma), \tag{7}$$

where $d\operatorname{vol}_p(A)$ is a Haar measure for $\mathrm{U}(p)$ and $d\operatorname{vol}_q(B)$ is the correspondingly normalized invariant measure for $\mathrm{Herm}^+(q)$. Invariance of $d\operatorname{vol}_q(B)$ means invariance under the transformation $B \mapsto gBg^\dagger$ for any $g \in \mathrm{GL}_q(\mathbb{C})$; the explicit expression for $d\operatorname{vol}_q(B)$ is

$$d\operatorname{vol}_q(B) = \frac{dB}{\operatorname{Det}^q(B)}, \quad dB \equiv \prod_{\lambda=1}^q dB^{\lambda\lambda} \prod_{1 \leq \nu < \nu' \leq q} 2\, d\mathfrak{Re}(B^{\nu\nu'})\, d\mathfrak{Im}(B^{\nu\nu'}).$$

The symbol $\partial_\tau \partial_\sigma$ is short for the product of partial derivatives

$$\partial_\tau \partial_\sigma = \prod_{\mu=1}^p \prod_{\nu=1}^q \frac{\partial^2}{\partial \tau^{\nu\mu}\, \partial \sigma^{\mu\nu}}.$$

We mention in passing that the Berezin form (7) is invariant under the action of a Lie superalgebra $\mathfrak{gl}(p|q) \times \mathfrak{gl}(p|q)$ where the first factor acts on the left of $Q$ and the second one on the right. This invariance property actually determines $DQ$ uniquely up to multiplication by a constant. The supermanifold of the supermatrix $Q$ belongs to one of the ten families of Riemannian symmetric superspaces described in [26]. We also note that in the important case of $p = q$ the Berezin form (7) is flat, i.e., is given by a product of differentials resp. partial derivatives:

$$DQ = (2\pi)^{-p^2} |dA|\, dB\, \partial_\tau \partial_\sigma. \tag{8}$$

The superdeterminant is the usual one, satisfying $\ln \mathrm{SDet}(Q) = \mathrm{STr} \ln Q$ with $\mathrm{STr}\, Q = \mathrm{Tr}\, B - \mathrm{Tr}\, A$ (bosons count as plus, fermions as minus):

$$\mathrm{SDet}^n(Q) = \frac{\mathrm{Det}^n(B)}{\mathrm{Det}^n(A - \sigma B^{-1}\tau)}. \tag{9}$$

We repeat once more that the inequality $n \geq q$ has to be satisfied in order for the formula (6) to be true.

For the case that $f$ is a Schwartz function (i.e., decreases faster than any power), a mathematical proof of the superbosonization formula (6) has been given in [25]. More precisely, the theorem stated and proved in [25] assumes (besides $n \geq q$) that $f$ is a *holomorphic* $\mathrm{U}(n)$-invariant function of the vectors $S_i$, $\bar{S}_i$, $\chi_i$, $\bar{\chi}_i$ $(i = 1, \ldots, n)$. These assumptions are always satisfied for the case of Gaussian disorder distributions.

In summary, compared to previous works which suggest the existence a formula such as (6), the main advance here is that the integration domain and sufficient conditions of validity have been specified. In the light of the discussion in the previous subsection, let us emphasize once more that there exist many choices of function $F$ (although in practical applications there will usually be a natural choice) so that $F(Q)$ becomes $f(\sum \psi_i \otimes \bar{\psi}_i)$ on substituting $Q \to \psi_i \otimes \bar{\psi}_i$. The theorem in [25] states that the superbosonization formula (6) holds true for *any* such choice of $F$ and that paper also gives analogous formulas for the cases of orthogonal and symplectic symmetry.

### 2.4 Proof of the Superbosonization Formula for $p = q = n$

We now elaborate on the case of $p = q = n$, which will be a good starting point for making the generalization to $n < q$ in Sect. 2.5. An application of the special case of $p = q = n = 1$ will be given in Sect. 3.

We simplify the notation by putting $r := p = q = n$. Thus we are now dealing with $r$ vectors $S_1, \ldots, S_r$ each of which has $r$ complex components, and with $r$ vectors $\chi_1, \ldots, \chi_r$ having $r$ anti-commuting components. Let $f$ be some function that depends only on the $\mathrm{U}(n)$-invariant combinations $(n = r)$

$$\sum_{i=1}^{r} \chi_i \otimes \bar{\chi}_i, \qquad \sum_{i=1}^{r} \chi_i \otimes \bar{S}_i, \qquad \sum_{i=1}^{r} S_i \otimes \bar{\chi}_i, \qquad \sum_{i=1}^{r} S_i \otimes \bar{S}_i,$$

viewed as the blocks of a supermatrix. We wish to compute the integral of such a function:

$$\int f \equiv (2\pi)^{-r^2} \int d\bar{S} dS\, \partial_{\bar{\chi}} \partial_{\chi}\, f \begin{pmatrix} \sum_i \chi_i \otimes \bar{\chi}_i & \sum_i \chi_i \otimes \bar{S}_i \\ \sum_i S_i \otimes \bar{\chi}_i & \sum_i S_i \otimes \bar{S}_i \end{pmatrix}. \tag{10}$$

For this purpose we choose some function $F(Q)$ of a supermatrix $Q$ so that making the substitution

$$Q \to \sum_i \psi_i \otimes \bar{\psi}_i = \begin{pmatrix} \sum_i \chi_i \otimes \bar{\chi}_i & \sum_i \chi_i \otimes \bar{S}_i \\ \sum_i S_i \otimes \bar{\chi}_i & \sum_i S_i \otimes \bar{S}_i \end{pmatrix},$$

we recover from $F$ the given function $f$. In the following we will change the integral $\int f$ in (10) to an integral over $F$ in four steps.

### 2.4.1 First Step

We begin by assuming that the vectors $S_1, \ldots, S_r$ are linearly independent, so that the $r \times r$ matrix which is formed by taking the vectors $S_1, \ldots, S_r$ to be the columns of that matrix is regular or of full rank. (The condition of linear independence is not always satisfied, of course. However, the sets of linearly dependent vectors form a set of measure zero. They can therefore be ignored for the purpose of integration.) Then we define square matrices $\sigma$ and $\tau$ of size $r \times r$ and with anti-commuting entries by

$$\sigma := \sum_{i=1}^{r} \chi_i \otimes \bar{S}_i, \qquad \tau := \sum_{i=1}^{r} S_i \otimes \bar{\chi}_i.$$

By the chain rule of differentiation we have

$$\partial_{\bar{\chi}} \partial_{\chi} = \mathrm{Det}^r \Big( \sum S_i \otimes \bar{S}_i \Big) \partial_\tau \partial_\sigma,$$

where $\partial_\tau \partial_\sigma \equiv \prod \partial^2 / \partial \tau^{\nu\mu} \partial \sigma^{\mu\nu}$. By the assumption that the vectors $S_1, \ldots, S_r$ are linearly independent, the $r \times r$ matrix

$$B := \sum_{i=1}^{r} S_i \otimes \bar{S}_i$$

has non-vanishing determinant and hence an inverse. Thus we can write

$$\sum_{i=1}^{r} \chi_i \otimes \bar{\chi}_i = \Big( \sum \chi_i \otimes \bar{S}_i \Big) \Big( \sum S_j \otimes \bar{S}_j \Big)^{-1} \Big( \sum S_k \otimes \bar{\chi}_k \Big) = \sigma B^{-1} \tau,$$

and our integral (10) then becomes

$$\int f = (2\pi)^{-r^2} \int d\bar{S} dS \, \mathrm{Det}^r(B) \partial_\tau \partial_\sigma F \begin{pmatrix} \sigma B^{-1} \tau & \sigma \\ \tau & B \end{pmatrix}.$$

### 2.4.2 Second Step

Now we take the Hermitian (and non-negative) matrix $B = \sum S_i \otimes \bar{S}_i$ as the new set of commuting variables of integration. Let $\delta(B)$ be the $\delta$-distribution (or $\delta$-function) centered at zero on the linear space of Hermitian $r \times r$ matrices $B$, and let $\delta(B - \sum S_i \otimes \bar{S}_i)$ be the shifted $\delta$-distribution. We then have

$$1 = \int dB \delta \Big( B - \sum S_i \otimes \bar{S}_i \Big),$$

where the integral is over the linear space of Hermitian $r \times r$ matrices $B$ with flat (or translation-invariant) measure $dB = \prod_k dB_{kk} \prod_{i<j} 2 d\mathfrak{Re}(B_{ij}) d\mathfrak{Im}(B_{ij})$. Now if $f_1$ is an (integrable) function of $\sum S_i \otimes \bar{S}_i$, then by inserting the relation above and changing the order of integration we get

$$\begin{aligned} \int d\bar{S} dS f_1 \Big( \sum S_i \otimes \bar{S}_i \Big) &= \int d\bar{S} dS \Big( \int dB \delta \Big( B - \sum S_i \otimes \bar{S}_i \Big) \Big) f_1 \Big( \sum S_i \otimes \bar{S}_i \Big) \\ &= \int dB f_1(B) \int d\bar{S} dS \, \delta \Big( B - \sum S_i \otimes \bar{S}_i \Big). \end{aligned}$$

To compute the integral $J(B) := \int d\bar{S}dS\delta\Big(B - \sum S_i \otimes \bar{S}_i\Big)$, let $g \in \mathrm{GL}_r(\mathbb{C})$ be any invertible $r \times r$ matrix with complex entries, and consider

$$J(gBg^\dagger) = \int d\bar{S}dS\delta\Big(gBg^\dagger - \sum S_i \otimes \bar{S}_i\Big).$$

Of course, since $\sum S_i \otimes \bar{S}_i \geq 0$, the function $J(B)$ vanishes if one or several eigenvalues of $B$ are negative. Changing variables from $S_i$ to $gS_i$ $(i = 1, \ldots, r)$ gives

$$J(gBg^\dagger) = \int d\bar{S}dS\,\mathrm{Det}^r(gg^\dagger)\delta\Big(g\Big(B - \sum S_i \otimes \bar{S}_i\Big)g^\dagger\Big).$$

Now the $\delta$-function obeys the transformation rule $\delta(gBg^\dagger) = \mathrm{Det}^{-r}(gg^\dagger)\delta(B)$. We therefore obtain $J(gBg^\dagger) = J(B)$. Since the action of $\mathrm{GL}_r(\mathbb{C})$ on the positive Hermitian matrices by $B \mapsto gBg^\dagger$ is transitive (i.e., every positive Hermitian matrix $B$ can be written in the form $B = gg^\dagger$), it follows that $J(B)$ is a constant independent of $B > 0$.

This constant is readily found by taking $B$ to be the $r \times r$ unit matrix, $B = \mathbf{1}_r$. Contributions to the integral

$$J(\mathbf{1}_r) = \int d\bar{S}dS\delta\Big(\mathbf{1}_r - \sum S_i \otimes \bar{S}_i\Big)$$

come only from those $S_1, \ldots, S_r$ that form an orthonormal system. Therefore $J(\mathbf{1}_r) = \mathrm{vol}\,\mathrm{U}(r)$. More precisely, if we make the factorization $(S_1, S_2, \ldots, S_r) = \mathrm{e}^X \mathrm{e}^{\mathrm{i}Y}$ with Hermitian $X$ and $Y$, then $d\bar{S}dS$ becomes $dBdk$ where $B = \mathrm{e}^{2X}$ and $dk$ for $k = \mathrm{e}^{\mathrm{i}Y}$ is the Haar measure for $\mathrm{U}(r)$ normalized by

$$dk|_{Y=0} = dY \equiv \prod_l dY_{ll} \prod_{i<j} 2\,d\mathfrak{Re}(Y_{ij})\,d\mathfrak{Im}(Y_{ij}).$$

Thus $J(\mathbf{1}_r) = \int dk \int dB\,\delta(\mathbf{1}_r - B) = \mathrm{vol}\,\mathrm{U}(r)$. After some of calculations (passing through the eigenvalue representation of $k$) one can show that with this choice of normalization the total volume of $\mathrm{U}(r)$ is

$$\mathrm{vol}\,\mathrm{U}(r) = \int_{\mathrm{U}(r)} dk = \frac{(2\pi)^{r(r+1)/2}}{0!1!\cdots(r-1)!}.$$

Altogether we obtain

$$\int d\bar{S}dS\, f_1\Big(\sum S_i \otimes \bar{S}_i\Big) = \mathrm{vol}\,\mathrm{U}(r) \int_{B>0} dB\, f_1(B).$$

Applying this relation to our situation gives

$$\int f = (2\pi)^{-r^2}\,\mathrm{vol}\,\mathrm{U}(r) \int_{B>0} dB\,\mathrm{Det}^r(B)\partial_\tau\partial_\sigma F\begin{pmatrix}\sigma B^{-1}\tau & \sigma\\ \tau & B\end{pmatrix}.$$

### *2.4.3 Third Step*

We now re-express the volume factor $\mathrm{vol}\,\mathrm{U}(r)$ as a dummy integral $\mathrm{vol}\,\mathrm{U}(r) = \int_{\mathrm{U}(r)} dk$. Having done so, we view $\mathrm{U}(r)$ as a real subgroup of the complex group $\mathrm{GL}_r(\mathbb{C})$, and we

extend the Haar measure $dk$ (regarded as a differential form) from $\mathrm{U}(r)$ to $\mathrm{GL}_r(\mathbb{C})$. For example, in the case of $r = 1$ this means that we set $k = \mathrm{e}^{\mathrm{i}y}$ with $y \in [0, 2\pi]$ and extend the given Haar measure $dy$ of the unit circle $\mathrm{U}(1) \subset \mathbb{C}$ to the holomorphic differential form $dy = (\mathrm{i}a)^{-1} da$ on $\mathrm{GL}_1(\mathbb{C})$ by letting $a = \mathrm{e}^{\mathrm{i}y}$ be any non-zero complex number.

In the general case, if $A_{ij}$ are the matrix elements of $A \in \mathrm{GL}_r(\mathbb{C})$, the Haar measure $dk$ extends from $\mathrm{U}(r)$ to $\mathrm{GL}_r(\mathbb{C})$ as the holomorphic differential form

$$dk = \mathrm{i}^{-r^2} \frac{dA}{\mathrm{Det}^r(A)}, \quad dA = \bigwedge_{i,j} dA_{i,j},$$

where the variables $A_{ij}$ for $i, j = 1, \ldots, r$ are regarded as a set of $r^2$ independent complex coordinates subject only to the condition $\mathrm{Det}(A) \neq 0$. (Mathematically speaking, by interpreting the Haar measure as a differential form we choose a fixed orientation of the unitary group.) To verify this formula for $dk$, notice that $\mathrm{Det}^{-r}(A)\, dA$ is invariant under the transformation $A \mapsto gAh$ for any $g, h \in \mathrm{GL}_r(\mathbb{C})$. This invariance property in fact determines the Haar measure $dk$ uniquely up to multiplication by a constant. To verify the normalization constant, one sets $A = \mathrm{e}^{\mathrm{i}Y}$ and notices that $\mathrm{Det}^{-r}(A)\, dA|_{Y=0} = \mathrm{i}^{r^2} dY$.

Let us now assume that $F\left(\begin{smallmatrix} \sigma B^{-1}\tau & \sigma \\ \tau & B \end{smallmatrix}\right)$ has been chosen as an analytic function of the left upper block. Then, using the fact that

$$\int_{\mathrm{U}(r)} dk k_{i_1,j_1} k_{i_2,j_2} \cdots k_{i_l,j_l} = 0$$

holds for any $l > 0$, we can write

$$\begin{aligned} \int f &= (2\pi)^{-r^2} \int_{B>0} dB\, \mathrm{Det}^r(B) \int_{\mathrm{U}(r)} dk \partial_\tau \partial_\sigma F \begin{pmatrix} k + \sigma B^{-1}\tau & \sigma \\ \tau & B \end{pmatrix} \\ &= (2\pi\mathrm{i})^{-r^2} \int_{B>0} dB\, \mathrm{Det}^r(B) \int_{\mathrm{U}(r)} dA\, \mathrm{Det}^{-r}(A) \partial_\tau \partial_\sigma F \begin{pmatrix} A + \sigma B^{-1}\tau & \sigma \\ \tau & B \end{pmatrix}. \end{aligned}$$

### 2.4.4 Fourth Step

In the final step, we use the fact that the holomorphic form $dA$ is invariant under translations. We exploit this invariance to make the (nilpotent) translation $A \to A - \sigma B^{-1}\tau$. Since the unitary group is a closed manifold (i.e., has no boundary), such a translation does not give rise to boundary terms. Therefore, the result after translation is

$$\int f = (2\pi\mathrm{i})^{-r^2} \int_{B>0} dB \int_{\mathrm{U}(r)} dA\, \partial_\tau \partial_\sigma \frac{\mathrm{Det}^r(B)}{\mathrm{Det}^r(A - \sigma B^{-1}\tau)} F \begin{pmatrix} A & \sigma \\ \tau & B \end{pmatrix}.$$

To bring this result into standard form, we write

$$DQ := (2\pi)^{-r^2} \left(\frac{dB}{\mathrm{Det}^r(B)}\right) \left(\frac{dA}{\mathrm{i}^{r^2}\, \mathrm{Det}^r(A)}\right) \partial_\tau \partial_\sigma\, \mathrm{Det}^r(A)\, \mathrm{Det}^r(B),$$

where the second factor (returning from differential forms to measures) agrees with our Haar measure $dk$ for $\mathrm{U}(r)$, and the first factor is a correspondingly normalized invariant measure

for $\mathrm{Herm}^+(r)$. These were denoted in the more general situation of Sect. 2.3 by $d\,\mathrm{vol}_p(A)$ and $d\,\mathrm{vol}_q(B)$ respectively. Note that from the relation

$$\mathrm{Det}^q(A-\sigma B^{-1}\tau)\,\mathrm{Det}^p(B-\tau A^{-1}\sigma)=\mathrm{Det}^r(A)\,\mathrm{Det}^r(B)$$

for $p=q=r$, the present expression for $DQ$ in fact coincides with (7).

Since the ratio of determinants $\mathrm{Det}(B)/\mathrm{Det}(A-\sigma B^{-1}\tau)$ equals the superdeterminant of the supermatrix $Q=\left(\begin{smallmatrix}A & \sigma\\ \tau & B\end{smallmatrix}\right)$, we now arrive at the desired formula

$$\int f=\int DQ\,\mathrm{SDet}^r(Q)F(Q),$$

where it is understood that we integrate over the unitary matrices $A\in\mathrm{U}(r)$ and the positive Hermitian $r\times r$ matrices $B\in\mathrm{Herm}^+(r)$. This completes the proof of the superbosonization formula (6) for $p=q=n=r$.

### 2.5 The Case of $n<q$

We now turn to the case of $p$ fermionic replicas and $q>n$ bosonic ones. This case is not covered by the results of [25] and needs separate treatment. As before, we wish to calculate the integral (5).

Let us first verify by inspection that formula (6) cannot be true in the present situation. For this purpose, let $p=q=r$ (but $r>n$) for simplicity, and consider some function $f$ that depends only on the combinations $\sum_{i=1}^n \chi_i\otimes\bar\chi_i$ and $\sum_{i=1}^n S_i\otimes\bar S_i$. After superbosonization the integrand $F$ is only a function of $A$ and $B$ (and does not depend on $\sigma$ and $\tau$). As we noted above, the invariant Berezin form (7) for $p=q=r$ is

$$DQ=(2\pi)^{-r^2}d\,\mathrm{vol}_r(A)d\,\mathrm{vol}_r(B)\partial_\tau\partial_\sigma\,\mathrm{Det}^r(A)\,\mathrm{Det}^r(B).$$

Also, using the expression (9) for the superdeterminant, our integrand becomes

$$\begin{aligned}DQ\,\mathrm{SDet}^n(Q)F(Q)=(2\pi)^{-r^2}d\,\mathrm{vol}_r(A)\;\mathrm{Det}^{r-n}(A)\,d\,\mathrm{vol}_r(B)\,\mathrm{Det}^{r+n}(B)\\ \times\,\partial_\tau\partial_\sigma\,\mathrm{Det}^n(\mathbf{1}_r-B^{-1}\tau A^{-1}\sigma).\end{aligned}$$

Now the Fermi integral

$$\partial_\tau\partial_\sigma\,\mathrm{Det}^n(\mathbf{1}_r-B^{-1}\tau A^{-1}\sigma)$$

vanishes identically, since $\mathrm{Det}^n(\mathbf{1}_r-B^{-1}\tau A^{-1}\sigma)$ can be at most of degree $n\times 2r$ in the matrix elements of $\sigma$ and $\tau$, whereas the differential operator $\partial_\tau\partial_\sigma$ is homogeneous of higher degree $2r^2>2nr$. (Thus there are not enough anti-commuting variables in the integrand to give a non-zero result when taking all partial derivatives.)

However, if we scale $A$ and $B$ out of $\mathrm{Det}^n(\mathbf{1}_r-B^{-1}\tau A^{-1}\sigma)$ by sending, say, $\sigma\to A\sigma$ and $\tau\to B\tau$, then the $B$-dependence of the integrand becomes

$$d\,\mathrm{vol}_r(B)\,\mathrm{Det}^n(B)F(\cdot,B)=dB\,\mathrm{Det}^{n-r}(B)F(\cdot,B),$$

which gives rise to a singularity when one (or several) of the eigenvalues of the positive matrix $B$ approach zero. This singularity is non-integrable if the integrand $F$ goes to a non-zero constant in the same limit. Thus the right-hand side of the superbosonization formula (6) is ill-defined (of the type of $0\times\infty$) in the present case. Based on the treatment given for $n=r$ in Sect. 2.4, we will now derive the correct formula for $n<r$.

*2.5.1 First Step*

We now have $n$ vectors $S_1, \ldots, S_n$, each with $q$ complex components. We arrange these as a $q \times n$ rectangular matrix $S := (S_1, \ldots, S_n)$ so that

$$SS^\dagger = \sum_{i=1}^{n} S_i \otimes \bar{S}_i.$$

We decompose the rectangular matrix $S$ as

$$S = \begin{pmatrix} a \\ b \end{pmatrix},$$

where $a$ is an $n \times n$ square matrix while the block $b$ is rectangular of size $(q-n) \times n$. For generic $S$ the square matrix $a$ is invertible and hence is an element $a \equiv g$ of the group $\mathrm{GL}_n(\mathbb{C})$. Defining $Z := ba^{-1}$ we then have

$$S = \begin{pmatrix} g \\ Zg \end{pmatrix}.$$

By simple power counting one sees that the volume element transforms as

$$d\bar{S}dS = d\bar{Z}dZ\, d\bar{g}dg\, \mathrm{Det}^{q-n}(gg^\dagger).$$

Here our notational conventions are the same as before, i.e.,

$$d\bar{g}dg := \prod_{i,j=1}^{n} 2\, d\mathfrak{Re}(g_{ij})\, d\mathfrak{Im}(g_{ij})$$

and the same expression goes for $d\bar{Z}dZ$. We now make a further change of variables $g \to (1+Z^\dagger Z)^{-1/2}h$, which results in

$$S = \begin{pmatrix} (1+Z^\dagger Z)^{-1/2}h \\ Z(1+Z^\dagger Z)^{-1/2}h \end{pmatrix}, \qquad d\bar{S}dS = \frac{d\bar{Z}dZ}{\mathrm{Det}^{q}(1+Z^\dagger Z)}\, d\bar{h}dh\, \mathrm{Det}^{q-n}(hh^\dagger).$$

To explain the above factorization of $S$ let $\mathrm{Mat}'_{q,n}(\mathbb{C})$ denote all complex $q \times n$ matrices with full rank $n$. Every $S \in \mathrm{Mat}'_{q,n}(\mathbb{C})$ can be decomposed as $S = Th$ with $h \in \mathrm{GL}_n(\mathbb{C})$ and $T \in \mathrm{Mat}'_{q,n}(\mathbb{C})$ being the truncation of a unitary $q \times q$ matrix to the first $n$ columns (i.e., the last $q-n$ columns are deleted). This decomposition is not unique. Indeed, if $S = Th$ is such a decomposition, then $S = (Tk^{-1})(kh)$ is also such a decomposition for any $k \in \mathrm{U}(n)$. Thus the correct mathematical statement of factorization is

$$\mathrm{Mat}'_{q,n}(\mathbb{C}) = (\mathrm{U}(q)/\mathrm{U}(q-n)) \times_{\mathrm{U}(n)} \mathrm{GL}_n(\mathbb{C}).$$

Locally—more precisely speaking: whenever $\mathrm{Det}^{-q}(1+Z^\dagger Z) \neq 0$—we can make a definite choice of $k$ by taking

$$T := \begin{pmatrix} (1+Z^\dagger Z)^{-1/2} \\ Z(1+Z^\dagger Z)^{-1/2} \end{pmatrix} \in \mathrm{Mat}'_{q,n}(\mathbb{C}).$$

The factorization $S = Th$ then means that we regard $S$ as being given by its regular part $h \in \mathrm{GL}_n(\mathbb{C})$ and a point of the Grassmann manifold $\mathrm{U}(q)/\mathrm{U}(q-n) \times \mathrm{U}(n)$, which is the set of realizations of $\mathbb{C}^n$ as a unitary subspace of $\mathbb{C}^q$.

From the standard fact that the metric tensor of $\mathrm{U}(q)/\mathrm{U}(n)\times\mathrm{U}(q-n)$ can be expressed by

$$\mathrm{Tr}\, dZ(\mathbf{1}_n + Z^\dagger Z)^{-1} dZ^\dagger (\mathbf{1}_{q-n} + ZZ^\dagger)^{-1},$$

one easily finds that $d\bar{Z}dZ\,\mathrm{Det}^{-q}(1+Z^\dagger Z)$ expresses the invariant measure of $\mathrm{U}(q)/\mathrm{U}(q-n)\times\mathrm{U}(n)$. We henceforth denotes this invariant measure by $d\,\mathrm{vol}(T)$. Thus we can summarize the discussion of this subsection by

$$S = Th, \quad d\bar{S}dS = d\,\mathrm{vol}(T)\, d\bar{h}dh\,\mathrm{Det}^{q-n}(hh^\dagger),$$

which results in the formula

$$\int f = (2\pi)^{-pn} \int d\,\mathrm{vol}(T) \int d\bar{h}dh\,\mathrm{Det}^{q-n}(hh^\dagger)\,\partial_{\bar{\chi}}\partial_\chi F \begin{pmatrix} \chi\bar{\chi} & \chi h^\dagger T^\dagger \\ Th\bar{\chi} & Thh^\dagger T^\dagger \end{pmatrix}.$$

Here $\chi$ is the $p\times n$ rectangular matrix $\chi = (\chi_1, \ldots, \chi_n)$, and $\bar{\chi}$ is the corresponding $n\times p$ rectangular matrix whose rows are the row vectors $\bar{\chi}_1, \ldots, \bar{\chi}_n$. In other words, we have $\sum \chi_i \otimes \bar{\chi}_i = \chi\bar{\chi}$, $\sum \chi_i \otimes \bar{S}_i = \chi S^\dagger$, etc.

### 2.5.2 Second Step

In the next step we take the matrix elements of $\sigma := \chi h^\dagger$ and $\tau := h\bar{\chi}$ as our new anticommuting variables. This gives (cf. Sect. 2.4.1)

$$\int f = (2\pi)^{-pn} \int d\,\mathrm{vol}(T) \int d\bar{h}dh\,\mathrm{Det}^{p+q-n}(hh^\dagger)\,\partial_\tau\partial_\sigma F \begin{pmatrix} \sigma(hh^\dagger)^{-1}\tau & \sigma T^\dagger \\ T\tau & Thh^\dagger T^\dagger \end{pmatrix}.$$

Then we make a change of (commuting) variables to $B := hh^\dagger$. By the same reasoning given in detail in Sect. 2.4.2, we obtain

$$\int f = (2\pi)^{-pn}\,\mathrm{vol}\,\mathrm{U}(n) \int d\,\mathrm{vol}(T) \int_{B>0} dB\,\mathrm{Det}^{p+q-n}(B)\,\partial_\tau\partial_\sigma F \begin{pmatrix} \sigma B^{-1}\tau & \sigma T^\dagger \\ T\tau & TBT^\dagger \end{pmatrix}.$$

The following steps are also similar to before. We introduce a dummy integral over $\mathrm{U}(p)$ with $(\mathrm{vol}\,\mathrm{U}(p))^{-1} \times \int_{\mathrm{U}(p)} \mathrm{Det}^{-p}(\mathrm{i}A)\,dA = 1$:

$$\int f = (2\pi)^{-pn} \frac{\mathrm{vol}\,\mathrm{U}(n)}{\mathrm{vol}\,\mathrm{U}(p)} \int d\,\mathrm{vol}(T) \int_{B>0} dB \int_{\mathrm{U}(p)} dA \frac{\mathrm{Det}^{p+q-n}(B)}{\mathrm{Det}^p(\mathrm{i}A)} \times \partial_\tau\partial_\sigma F \begin{pmatrix} A+\sigma B^{-1}\tau & \sigma T^\dagger \\ T\tau & TBT^\dagger \end{pmatrix}.$$

Then we make the shift $A \to A - \sigma B^{-1}\tau$. Expressing the result in terms of the invariant measures $d\,\mathrm{vol}_p(A) = \mathrm{Det}^{-p}(\mathrm{i}A)\,dA$ and $d\,\mathrm{vol}_n(B) = \mathrm{Det}^{-n}(B)\,dB$, we obtain

$$\int f = (2\pi)^{-pn} \frac{\mathrm{vol}\,\mathrm{U}(n)}{\mathrm{vol}\,\mathrm{U}(p)} \int d\,\mathrm{vol}(T) \int_{B>0} d\,\mathrm{vol}_n(B) \times \int_{\mathrm{U}(p)} d\,\mathrm{vol}_p(A)\,\partial_\tau\partial_\sigma \frac{\mathrm{Det}^q(B)}{\mathrm{Det}^{-p}(B-\tau A^{-1}\sigma)} F \begin{pmatrix} A & \sigma T^\dagger \\ T\tau & TBT^\dagger \end{pmatrix}. \tag{11}$$

This formula is arguably more complicated than (6). The reason for this is that the rank of $\sum_{i=1}^{n} S_i \otimes \bar{S}_i$ never exceeds $n$, so there exist at least $q-n$ zero eigenvalues and the range of $\sum S_i \otimes \bar{S}_i$ is a *submanifold of the boundary* of $\mathrm{Herm}^+(q)$. Our derivation parameterizes this submanifold by two factors. The first factor, $T$, describes the $n$-dimensional complement of the kernel space of $\sum S_i \otimes \bar{S}_i$ in $\mathbb{C}^q$; thus it describes the complex $n$-plane spanned by the vectors $S_1, \ldots, S_n$ in $\mathbb{C}^q$. (The set of such subspaces $\mathbb{C}^n \hookrightarrow \mathbb{C}^q$ is in one-to-one correspondence with points of the symmetric space $\mathrm{U}(q)/\mathrm{U}(n) \times \mathrm{U}(q-n)$). The second factor, $B$, is the operator $\sum S_i \otimes \bar{S}_i$ restricted to its complex $n$-plane $\mathbb{C}^n$.

The merit of the result (11) is it that allows us to make an exact transformation of the original problem to supermatrix variables in the parameter range $n < q$. Unfortunately, the invariance properties of the integral on the right-hand side are not very transparent. In other words, the Lie superalgebra $\mathfrak{gl}(p|q) \times \mathfrak{gl}(p|q)$ acts as first-order differential operators on the functions $f$ and $F$, and while the transformation behavior with respect to this action is very clear on the left-hand side, it is not easy to see how the desired behavior emerges on the right-hand side.

### 2.6 Supermatrix Model

Let us now return to the formulation of Sect. 2.3, which is valid for $n \geq q$. The superbosonization formula (6) allows us to replace the initial problem of computing the correlation functions for a Gaussian ensemble of Hermitian random matrices with variance matrix $C_{ij}$, (2), with the problem of computing the generating function

$$
\begin{aligned}
Z[J] &= \int \prod_{i=1}^{M} DQ_i \exp(-F[Q]), \quad F = F_0 + F_1, \\
F_0[Q] &= \frac{1}{2} \sum_{i,j=1}^{M} C_{ij} \,\mathrm{STr}\, Q_i s Q_j s + n \sum_{i=1}^{M} \mathrm{STr} \ln Q_i - \mathrm{i}E \sum_{i=1}^{M} \mathrm{STr}\, Q_i s, \\
F_1[Q] &= -\mathrm{i} \sum_{i=1}^{M} \mathrm{STr}\, Q_i s(\mathcal{E} - E \cdot \mathbf{1} + J_i),
\end{aligned}
\tag{12}
$$

where $E$ is the "center of mass" of the energy parameters in $\mathcal{E}$. This is the most general supermatrix model for ensembles of Gaussian random matrices with unitary symmetry $\mathrm{U}(n)^M$ ($M = N/n$). Note that no approximations have been made, and (12) is an *exact* reformulation of the original problem. The entries of the variance matrix $C_{ij}$ are required to be real symmetric and positive, but are otherwise arbitrary. The size of the supermatrices $Q$ depends on the correlation function to be calculated and is the same as for the supermatrices $Q$ of the standard non-linear sigma model [1, 2]. However, while the eigenvalues of the usual sigma model field $Q$ are constrained to be $\pm 1$, the eigenvalues of our superbosonization field $Q_i$ fluctuate; they are real and positive in the boson-boson sector and unitary (i.e., of unit modulus) in the fermion-fermion sector.

Let us also point out that, since the key formula (6) holds for a large class of functions $f$ (not just the Gaussian functions), the present method is not restricted to Gaussian disorder distributions. In the case of a more general disorder distribution, the term $\exp(-\frac{1}{2}\sum_{i,j=1}^{M} C_{ij}\,\mathrm{STr}\, Q_i s Q_j s)$ in (12) is replaced by a functional which is determined by the Fourier transform of the disorder distribution function.

It should also be clear that such a description as (12) exists even for $n < q$. We just need to replace the integration domain and integration measure $DQ_i\ \mathrm{SDet}^n(Q_i)$ by the modified one constructed in Sect. 2.5.

## 3 Density of States for Almost Diagonal Matrices

We now demonstrate how the method developed in the previous section works for the density of states of almost diagonal random Hermitian matrices. We focus on the $n = 1$ case, which was investigated in [18], and we will compare the results of this reference with ours. Note that the limit of almost diagonal random matrices is not accessible (for $n = 1$) via the standard non-linear sigma model for random matrix problems.

To calculate the density of states from the generating function (12), we set $p = q = 1$ and $s = \mathbf{1}$. Our superbosonization field $Q_i$ is now a $2 \times 2$ supermatrix

$$Q_i = \begin{pmatrix} a_i & \sigma_i \\ \tau_i & b_i \end{pmatrix}$$

with Berezin integral form

$$DQ_i = (2\pi \mathrm{i})^{-1}\, db_i\, da_i\, \partial_{\tau_i} \partial_{\sigma_i}.$$

The density of states per unit length for a system of $N$ sites is expressed as

$$\rho(E) = (2\pi N)^{-1} \Im\mathrm{m} \sum_{i=1}^{N} \mathrm{i} \langle \mathrm{Tr}\, Q_i \rangle, \tag{13}$$

where $\langle \ldots \rangle$ means the average with respect to the statistical weight $\mathrm{e}^{-F[Q]}$ of (12), taken with vanishing source term $J_i = 0$. The diagonal matrix of energy parameters is $\mathcal{E} = \mathrm{diag}(E, E)$ with $\Im\mathrm{m}\, E > 0$.

We now proceed by first solving the diagonal variance matrix $C_{ij} = C_0\, \delta_{ij}$ case exactly and then expanding in the off-diagonal terms $C_{ij}$, $i \neq j$. For notational simplicity we assume that the system is one-dimensional with translation-invariant $C_{ij} = c(|i - j|)$ (although neither assumption is really necessary).

In the zeroth order expansion the integral $\frac{1}{2}\langle \mathrm{Tr}\, Q_i \rangle = \langle b_i \rangle = \langle a_i \rangle$ in (13) factors as a product of $N$ independent integrals, one for each site. The $N - 1$ integrals for the sites $j \neq i$ all are unity:

$$\int DQ_j\, \mathrm{SDet}(Q_j) \mathrm{e}^{\mathrm{i}E\, \mathrm{STr}\, Q_j - (C_0/2)\, \mathrm{STr}\, Q_j Q_j} = 1 \quad (C_0 = C_{jj}),$$

which is a consequence of supersymmetry. The remaining integral for the distinguished site $i$ is

$$\begin{aligned} &\int DQ_i\, \mathrm{SDet}(Q_i) a_i \mathrm{e}^{\mathrm{i}E\, \mathrm{STr}\, Q_i - (C_0/2)\, \mathrm{STr}\, Q_i Q_i} \\ &\quad = (2\pi \mathrm{i})^{-1} \int_{\mathbb{R}_+} db \oint_{\mathrm{U}(1)} da\, \partial_\tau \partial_\sigma (b - \tau a^{-1} \sigma)\, \mathrm{e}^{\mathrm{i}E(b-a) - (C_0/2)(b^2 - a^2 - 2\sigma\tau)} \\ &\quad = \int_0^\infty \mathrm{e}^{\mathrm{i}Eb - (C_0/2)b^2} db, \end{aligned} \tag{14}$$

where in the second line we dropped the index $i$ from the integration variables. The $\sigma\tau$ term in the exponent cannot contribute to the Fermi integral $\partial_\tau \partial_\sigma$, as one must pick the term $-\tau a^{-1} \sigma$ in front of the exponential in order to have a non-zero integral over $a \in \mathrm{U}(1)$.

Hence, denoting the density of states in the zeroth order of this expansion by $\rho_d(E)$, we have (for $\Im\mathrm{m}\, E \to 0+$)

$$\rho_d(E) = \pi^{-1}\Im\mathrm{m}\langle a_1\rangle = (2\pi)^{-1}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}Eb-(C_0/2)b^2}db = (2\pi C_0)^{-1/2}\mathrm{e}^{-E^2/2C_0}. \tag{15}$$

This Gaussian function is of course none other than the probability distribution function of the diagonal elements of our almost diagonal random matrix.

Now we calculate the correction $\delta\rho_d(E)$ coming from the off-diagonal elements. For an almost diagonal variance matrix $C_{ij}$ we may approximate the generating function $Z[J]$ by expanding the exponential in the off-diagonal terms:

$$Z[J] = \int \prod_{i=1}^{N} DQ_i\ \mathrm{SDet}(Q_i)\mathrm{e}^{\mathrm{i}\,\mathrm{STr}\,Q_i(\mathcal{E}+J_i)-(C_0/2)\,\mathrm{STr}\,Q_iQ_i}\left(1-\frac{1}{2}\sum_{j\neq k}C_{jk}\,\mathrm{STr}\,Q_jQ_k+\cdots\right).$$

Therefore the correction to $\rho_d(E)$ may be written as

$$\delta\rho(E) = -(4\pi N)^{-1}\Im\mathrm{m}\,\mathrm{i}\sum_i\sum_{j\neq k}C_{jk}\langle\mathrm{Tr}(Q_i)\,\mathrm{STr}(Q_jQ_k)\rangle. \tag{16}$$

Again, $\langle\cdots\rangle$ is a product of $N$ independent integrals, $N-3$ of which are unity by supersymmetry. Using the relation $\langle Q^{\mu\nu}\rangle = \frac{1}{2}\langle\mathrm{Tr}\,Q\rangle\delta^{\mu\nu}$ and $\mathrm{STr}\,\mathbf{1}=0$, we see that $\langle\cdots\rangle$ in (16) vanishes unless $i=j$ or $i=k$. If $i=k\neq j$ then

$$\langle\mathrm{Tr}(Q_i)\,\mathrm{STr}(Q_jQ_i)\rangle = \frac{1}{2}\langle\mathrm{Tr}(Q_i)\,\mathrm{STr}(Q_i)\rangle\langle\mathrm{Tr}(Q_j)\rangle.$$

The single-site average of $\mathrm{Tr}\,Q_j$ is given by (14). The expression for the single-site average of $\mathrm{Tr}(Q_i)\,\mathrm{STr}(Q_i)$ is the same except that an additional factor $\mathrm{STr}\,Q_i = b_i - a_i$ has to be inserted under the integral sign. The term $-a_i$ of this factor gives vanishing residue at the simple pole $a_i=0$ and hence does not contribute. So only an extra factor of $b_i\equiv b$ remains and we get

$$\int DQ_i\,\mathrm{SDet}(Q_i)\,\mathrm{Tr}(Q_i)\,\mathrm{STr}(Q_i)\mathrm{e}^{\mathrm{i}E\,\mathrm{STr}\,Q_i-(C_0/2)\,\mathrm{STr}\,Q_iQ_i} = \int_0^\infty \mathrm{e}^{\mathrm{i}Eb-(C_0/2)b^2}b\,db.$$

Altogether we then obtain for the correction to $\rho_d(E)$ (when the variance matrix $C_{ij}$ is almost diagonal):

$$\delta\rho(E) = -\frac{C_1}{2C_0}\frac{d}{dE}(\mathrm{e}^{-E^2/C_0}\mathrm{erfi}(E/\sqrt{2C_0})), \tag{17}$$

where $C_1 = \sum_{j\neq 1}C_{1j}$ and $\mathrm{erfi}(x)$ is the imaginary error function,

$$\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!(2n+1)}.$$

Note that $\delta\rho(E)$ is a total derivative. This is as expected as turning on the off-diagonal elements of $C_{ij}$ neither creates nor destroys levels but just changes their positions.

The high-energy limit of the density of states for almost diagonal random matrices was calculated in Ref. [18]. To compare with those results, we note that the large-$x$ limit of the

imaginary error function is $xe^{-x^2}\mathrm{erfi}(x) \to \sqrt{\pi}$, which gives

$$\delta\rho(E) = \frac{C_1}{C_0}\rho_d(E) \quad (E \gg \sqrt{2C_0}).$$

Thus we have the simple result that the density of states for an almost diagonal Gaussian random matrix at high energies (i.e., in the far tail of the Gaussian distribution) is

$$\rho(E) = \sum_j (C_{ij}/C_{ii})\rho_d(E) \quad (E \gg \sqrt{2C_{ii}}). \tag{18}$$

This result is equivalent to that calculated in Ref. [18].

## 4 Reduction to the Standard Sigma Model

Let us now see how the generalized supermatrix model (12) reduces to the standard diffusive non-linear sigma model under suitable conditions. In order for such a reduction to take place, the superbosonization field must get localized in a certain low-energy submanifold (corresponding to the standard sigma model field) of the total field space. As we shall see, the latter comes about if the number of orbitals $n$ is large. If, in addition, the superbosonization field has enough collectivity (or stiffness) due to a variance matrix $C_{ij}$ with sufficiently long range, then the effective degrees of freedom of the problem are the Goldstone modes associated with the low-energy manifold, and one recovers the diffusive non-linear sigma model.

To identify the low-energy manifold, we must first understand the symmetries of the functional $F_0[Q]$ in (12). Beginning with the boson-boson sector, let $g \in \mathrm{GL}_q(\mathbb{C})$ be any invertible complex $q \times q$ matrix and consider the transformation

$$A_i \mapsto A_i, \qquad B_i \mapsto g B_i g^\dagger, \qquad \sigma_i \mapsto \sigma_i g^\dagger, \qquad \tau_i \to g\tau_i.$$

This transformation is a symmetry of the Berezin integral form $DQ_i$ of (7). It is a symmetry of $F_0[Q]$ if

$$g^\dagger s_0 g = s_0, \quad s_0 = \begin{pmatrix} \mathbf{1}_{q_+} & 0 \\ 0 & -\mathbf{1}_{q_-} \end{pmatrix}, \tag{19}$$

where $s_0$ is the boson-boson part of $s$. The condition (19) singles out a pseudo-unitary subgroup of $\mathrm{GL}_q(\mathbb{C})$. This non-compact group is denoted by $\mathrm{U}(q_+, q_-)$.

Turning to the fermion-fermion sector, let $(g, h) \in \mathrm{U}(p) \times \mathrm{U}(p)$ and consider the transformation

$$A_i \mapsto g A_i h^{-1}, \qquad B_i \mapsto B_i, \qquad \sigma_i \mapsto g\sigma_i, \qquad \tau_i \to \tau_i h^{-1}.$$

Again this is a symmetry of $DQ_i$. In order for it to be a symmetry of $F_0[Q]$ we must impose the condition $g = h$, which singles out the diagonal subgroup $\mathrm{U}(p) \subset \mathrm{U}(p) \times \mathrm{U}(p)$.

The product of groups $\mathrm{U}(p) \times \mathrm{U}(q_+, q_-)$ is the group of bosonic symmetries of our problem. There also exist a number of fermionic symmetries. It is not difficult to show that on inclusion of these symmetries the symmetry group becomes a Lie supergroup $\mathrm{U}(p \,|\, q_+, q_-)$. This is known to be the symmetry group of the standard non-linear sigma model (for systems in the unitary symmetry class). Note that the group action of $\mathrm{U}(p \,|\, q_+, q_-)$ is by conjugation $Q_i s \mapsto U Q_i s U^{-1}$.

The next step is to look for minima of the 'energy' functional $F_0[Q]$ in (12). (The term $F_1$ is considered small for present purposes). Varying this functional gives the saddle-point equation

$$0 = \frac{\delta}{\delta Q_i} F_0[Q] = \sum_j C_{ij} s\, Q_j s - (Q_i)^{-1} n - \mathrm{i}Es. \tag{20}$$

Assuming translation invariance of $C_{ij}$ (and setting $C := C_0 + C_1 = \sum_j C_{ij}$) one first looks for $i$-independent solutions $Q_i \equiv Q$ in the space of diagonal matrices. From (20) each eigenvalue of the diagonal matrix $Q$ satisfies a quadratic equation, which in general has two different solutions, $q^{(+)}$ and $q^{(-)}$. In the boson-boson sector, one of these solutions, say $q^{(-)}$, is ruled out by the positivity condition $B_i > 0$. However, in the fermion-fermion sector both solutions $q^{(\pm)}$ are in principle admissible. Connected supermanifolds of solutions of the saddle-point equation are then generated by the conjugation action $Qs \mapsto U\,Qs\,U^{-1}$ of the symmetry group. The saddle-point manifolds thus obtained are orbits of $\mathrm{U}(p\,|\,q_+, q_-)$ which can be classified by the number of eigenvalues $q^{(+)}$ and $q^{(-)}$ of the solution in the fermion-fermion sector. For energies 'inside the band', $|E| < \sqrt{4Cn}$, all these supermanifolds are parameterized by

$$Qs = C^{-1/2}(n - E^2/4C)^{1/2} U \Lambda U^{-1} + \mathrm{i}(E/2C)\,\mathbf{1}, \quad U \in \mathrm{U}(p\,|\,q_+, q_-), \tag{21}$$

where $\Lambda$ is a diagonal matrix with eigenvalues $\pm 1$:

$$\Lambda = \mathrm{diag}(s_1, s_0).$$

The boson-boson part of $\Lambda$ is uniquely determined by the boson-boson part $s_0$ of $s$, but the signs in the fermion-fermion part $s_1$ of $\Lambda$ are arbitrary. In the general situation, each of these saddle-point manifolds (corresponding to different choices of $\mathrm{Tr}\, s_1$) make a contribution to the generating function (12).

In the following we focus on the important case $p = q$. By making the maximally supersymmetric choice $s_1 = s_0$, one obtains a distinguished saddle-point manifold which dominates (under conditions to be specified), while contributions from other saddle-point manifolds are suppressed by fermionic zero modes due to a breaking of supersymmetry. Low-energy configurations of the superbosonization field are now expressed as

$$Q_i s = \pi \nu\, \tilde{Q}_i + \mathrm{i}(E/2C)\,\mathbf{1}, \quad \nu = \pi^{-1} C^{-1/2}(n - E^2/4C)^{1/2}, \tag{22}$$

where $\tilde{Q}_i$ is a dimensionless field, namely the standard sigma model field

$$\tilde{Q}_i = U_i \Lambda U_i^{-1}, \quad \Lambda = \mathrm{diag}(s_0, s_0),\ U_i \in \mathrm{U}(p\,|\,p_+, p_-).$$

By computing the second variation of $F_0[Q]$ at the minimum, one sees that the fluctuations of modes transverse to the low-energy manifold (the so-called massive modes) are controlled by the quadratic form

$$h_{ij} = (\pi\nu)^2 C_{ij} + n\delta_{ij} \overset{E=0}{=} n(\delta_{ij} + C_{ij}/C).$$

(Here the second expression makes the simplifying assumption that $E = 0$.) Since this quadratic form is bounded from below by the diagonal form $n\delta_{ij}$, its eigenvalues are never smaller than $n$ and it therefore follows that fluctuations of the massive modes are strongly suppressed in the limit of $n \gg 1$.

In the case of small $n$, when $C_{ij}$ is short ranged (i.e., our random matrix is almost diagonal), the massive modes fluctuate strongly and there is no controlled reduction to the low-energy manifold of fields $\tilde{Q}_i$. One might now think that the situation gets better when $C_{ij}$ is taken to be long ranged. However, this is definitely *not* true when the superbosonization method is used! The problem is that the Fourier transform $\tilde{C}(k)$ of a long-ranged variance matrix $C_{ij} = c(|i-j|)$ decreases with increasing wave number $k$ and is close to zero at high wave numbers (near the edge of the Brillouin zone), which implies that our high-momentum massive modes always have a small mass (of order $n$) when $n$ is small.

It should be stressed that the case of a long-ranged variance matrix is much better handled by the traditional Hubbard–Stratonovich transformation approach. There, fluctuations of the massive modes (say at $E = 0$) are controlled by another quadratic form,

$$h_{ij}^{(\mathrm{HS})} = n(\delta_{ij} + C w_{ij}),$$

where $w_{ij}$ is the matrix inverse of $C_{ij}$ (i.e., $\sum_j w_{ij} C_{jl} = \delta_{il}$). The Fourier transforms of $C_{ij}$ and $w_{ij}$ are reciprocals of each other, $\tilde{w}(k) = \tilde{C}(k)^{-1}$. Thus, if the eigenvalues of $C_{ij}$ become small near the edge of the Brillouin zone, then those of $w_{ij}$ become large, and therefore the high-momentum massive modes of the Hubbard–Stratonovich approach are truly massive for the case of long-ranged $C_{ij}$ (and all $n$ including $n = 1$) and can be integrated out in a controlled way. In summary, since a different quadratic form enters the game, the reduction to the standard sigma model in the present approach is valid under conditions that are not identical to those of the Hubbard–Stratonovich approach.

We now insert (22) into the expression (12) for $F_0$ to obtain an effective energy functional for the $\tilde{Q}_i$ field:

$$F_0 \approx \frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \,\mathrm{STr}\, \tilde{Q}_i \tilde{Q}_j, \quad \tilde{C}_{ij} = (\pi \nu)^2 C_{ij}.$$

This is a good approximation when $n \gg 1$, in which case we may simply neglect the massive modes. On the other hand, if $n$ is small the massive modes fluctuate strongly and we face the non-trivial task of integrating them out with a non-perturbative calculation. While this will not change the symmetries of the effective energy functional for the $\tilde{Q}_i$ field, it may cause a major renormalization of the coupling parameters $\tilde{C}_{ij}$.

Therefore, to maintain quantitative control of the situation, we now assume $n$ to be large. Taking into account the term $F_1$ in (12), we arrive at a low-energy effective action

$$\tilde{F}[\tilde{Q}] = \frac{1}{2} \sum_{i \neq j} \tilde{C}_{ij} \,\mathrm{STr}\, \tilde{Q}_i \tilde{Q}_j - \mathrm{i}\pi\nu \sum_{i=1}^{M} \mathrm{STr}\, \tilde{Q}_i (\mathcal{E} + J_i), \tag{23}$$

where diagonal terms have been dropped from the double sum over $i, j$ because $\mathrm{STr}\, \tilde{Q}_i \tilde{Q}_i = 0$. Recall that $\tilde{Q}_i$ is the standard sigma model field.

The action (23) describes a spectrum of physical situations ranging from strong localization to diffusive behavior. Strong localization occurs in the limit of almost diagonal random matrices, where our coupling coefficients $\tilde{C}_{ij}$ $(i \neq j)$ are small, leading to a disordered sigma model field $\tilde{Q}_i$. (Note that small $\tilde{C}_{ij}$ can be realized in spite of $n$ being large.)

The diffusive regime can be realized by taking the variance matrix $C_{ij}$ to be long ranged. In this limit, spatial variations of $\tilde{Q}_i$ are suppressed and we may make a continuum approximation $\tilde{Q}_i \to \tilde{Q}(r)$, expanding the first term on the right-hand side of (23) in gradients of

$\tilde{Q}(r)$. A standard computation gives the diffusive action

$$\frac{1}{2}\sum_{i\neq j}\tilde{C}_{ij}\,\mathrm{STr}\,\tilde{Q}_i\tilde{Q}_j \approx -\frac{\pi\nu}{8}\,\mathrm{STr}\int dr\,D\big(\nabla\tilde{Q}(r)\big)^2,$$

where the 'diffusion coefficient' is $D = 2\pi\nu\sum_j |i-j|^2 C_{ij}$. To summarize, the discussion above has shown how, in some cases, the conventional non-linear sigma model can be obtained by reduction from the supermatrix model (12).

Now in various regimes such a reduction cannot be done and one has to work with (12). Among these are models with a critical point where an Anderson metal-insulator transition takes place. Much attention has been given to such models [15, 18, 27, 28] with the aim of better understanding the critical behavior at the transition. It is an interesting open question whether the supermatrix model (12) can be of use in the analysis of these critical models.

## 5 Conclusion

The main outcome of this paper is an exact mapping of a Gaussian random matrix problem to a supermatrix model. The measure of integration and the structure of the matrix field $i \mapsto Q_i$ are given in (7) for $n \geq q$ and in (11) for $n < q$. One might be tempted to call the supermatrix model a 'generalized non-linear sigma model', but such a terminology would be misleading as the target space of the supermatrix model is not homogeneous (unlike with what is called a non-linear sigma model in the strict sense of the word) with respect to its symmetry group.

A notable result is that the superbosonization formulas (6) and (11) are *different*, depending on the value of the number of 'orbitals' $n$ relative to the number of commuting variables $q$ in the field $\psi$. The former determines the *real space* structure of the random matrix ensemble, while the latter is equal to the number of points $q$ in the $q$-point correlation function in the *energy space*. Thus the superbosonization formulas (6) and (11) indicate that there exists a correlation, in a certain sense, between real space and energy space. This fact seems to be fundamental, although its implications are not yet fully understood.

In this context, let us make one observation which we consider to be of relevance. It has been a puzzle for many years now how an inspired use of the replica trick with fermionic replicas correctly reproduces [29–32] the DoS-DoS correlation function in the large-$n$ random matrix limit, whereas the replica trick with bosonic replicas is known to fail [33] when the same limit is invoked. Based on the difference between $q \leq n$ and $q > n$ observed above, we propose the following resolution of this long-standing puzzle. (See also [34] for some recent speculation related to this issue.)

In the replica trick one needs to calculate the observable of interest for every number of replicas and then, using all this information, one tries to find an analytic continuation to zero replica number. When using the replica trick with $q$ bosonic replicas, we conjecture that vital information about the analytic continuation $q \to 0$ is contained in the *high* replica numbers $q > n$, where the behavior of the correlation functions seems to be qualitatively different (based on what we have seen with the superbosonization formula) from that for $q \leq n$. From this vantage point, it has to be regarded as an ill-advised scheme to take the large-$n$ limit (and thus the saddle-point approximation leading to the bosonic replica NL$\sigma$M) at an early stage of the calculation, as was done in [33]. Indeed, in the process of taking $n \to \infty$ the $q > n$ branch, and hence all information carried by it, is lost from the computation.

This scenario and other questions related to the difference between the cases $n \geq q$ and $n < q$ are interesting directions for future research. Here, in contrast, we outline the idea of a *uniform* representation of both cases which requires a non-trivial limiting procedure. For simplicity we restrict ourselves to the case of $n = 1$, $p = q = 2$. As was explained in detail at the beginning of Sect. 2.5, the simple superbosonization formula (6) fails in this case. For the sake of discussion let us recall the details here. The argument is especially transparent when the function $F(Q)$ depends only on the commuting variables in the matrices $A$ and $B$ and is non-zero in the limit $B \to 0$. Note that the left-hand side of (6) is non-zero in general. On the other hand, since $A$, $B$, $\sigma$, and $\tau$ for $p = q = 2$ are $2 \times 2$ matrices, the superdeterminant

$$\mathrm{SDet}(Q) = \mathrm{Det}(B)\,\mathrm{Det}^{-1}(A)\,\mathrm{Det}(1 - A^{-1}\sigma B^{-1}\tau)$$

is a quartic polynomial in eight Grassmann variables $\sigma^{\mu\nu}$ and $\tau^{\nu\mu}$ ($\mu, \nu = 1, 2$) and therefore $\prod \partial_{\tau^{\nu\mu}} \partial_{\sigma^{\mu\nu}} \mathrm{SDet}(Q) = 0$. This means that the right-hand side of (6) is zero, unless there is a divergence in the integration over commuting variables. Here one must pay attention to the fact that the integral over the bosonic variables $B$ can be singular along the $\mathrm{Det}(B) = 0$ boundary. A quick way to see this is to make a transformation to new variables $\sigma' = A^{-1}\sigma$ and $\tau' = B^{-1}\tau$, which has Jacobian $\mathrm{Det}^{-q}(A)\,\mathrm{Det}^{-q}(B)$ and leads to a singular dependence $\mathrm{Det}^{1-q}(B) = \mathrm{Det}^{-1}(B)$ at $q = 2$. Thus, a naive application of the superbosonization formula (6) to the case $n = 1$, $p = q = 2$ leads to an ill-defined expression of the type $0 \times \infty$.

The idea of a unified description stems from the observation that the deficiency of the Grassmann variables in $\mathrm{SDet}(Q)$ (where 4 are missing from the full set of 8), which is responsible for the 0 part in $0 \times \infty$, does not persist when $\mathrm{SDet}(Q)$ is replaced by $\mathrm{SDet}^{1+\alpha}(Q)$ with $\alpha \in \mathbb{C}$ being a small regularization parameter. Of course, to move $\alpha$ off zero, we must first define the factor $\mathrm{Det}^{-1-\alpha}(A)$ in $\mathrm{SDet}^{1+\alpha}(Q)$. This can be done by making some choice of fundamental domain for the logarithm $\ln : \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}$. Restricting $A \in \mathrm{U}(2) \subset \mathrm{GL}_2(\mathbb{C})$ to this fundamental domain, we may expand

$$\mathrm{SDet}^{1+\alpha}(Q) = \mathrm{SDet}(Q)(1 + \alpha \ln \mathrm{SDet}(Q) + \cdots).$$

One then easily verifies that the Fermi integral of $\mathrm{SDet}^{1+\alpha}(Q)$ is non-zero for small but non-zero $\alpha$. At the same time, the singular factor $\mathrm{Det}^{-1}(B)$ in the integrand is replaced by $\mathrm{Det}^{\alpha-1}(B)$, which makes the singularity at $B \to 0$ integrable for $\mathfrak{Re}\,\alpha > 0$. Moreover, it is possible to analytically continue the integral over $A \in \mathrm{U}(2)$ to $\alpha \neq 0$ by taking the integration contour, i.e., the 4-dimensional real submanifold of integration in $\mathrm{GL}_2(\mathbb{C})$, to infinity along the boundary of the fundamental domain of the logarithm. (This will work if the boundary of the fundamental domain has been chosen as a submanifold of rapid decay of the integrand.) Thus, carefully taking the limit $\lim_{\alpha \to 0+} \mathrm{SDet}^{1+\alpha}(Q)$ instead of immediately setting $\alpha = 0$, is a way to give meaning to the ill-defined expression $0 \times \infty$.

Exploratory calculations done along these lines give a remarkable coincidence with (11). Our explorations suggest that the superbosonization formula in both the $n = q = 1$ and $n = 1, q = 2$ cases can be written in a uniform way resembling the original form of Ref. [20] and (6) but with $\lim_{\alpha \to 0+} \mathrm{SDet}^{1+\alpha}(Q)$ replacing $\mathrm{SDet}(Q)$. It should be stressed, however, that the mathematics at hand gets more complicated as the number of replicas $p = q$ goes up. We have not yet made a serious effort to confront these complications and more work is needed to put the idea of $\alpha$-regularization on a solid mathematical basis in the general case.

To finish this Conclusion, we would like to compare our superbosonization (SB) approach with the standard one [1, 2] based on the Hubbard–Stratonovich (HS) transformation and the degenerate saddle-point approximation. As was mentioned in the Introduction,

the main disadvantage of the standard approach is that it does not apply to random matrix ensembles with a short-ranged variance matrix $C_{ij}$. Another disadvantage is that it is applicable only to Gaussian ensembles. Both restrictions are lifted in the new SB approach. The new approach is in some sense dual to the standard HS approach as it relies on the variance matrix $C_{ij}$, while the HS approach relies on its *inverse*, $(C^{-1})_{ij}$. This feature makes the new SB approach most efficient for short-ranged $C_{ij}$, e.g., for almost diagonal random matrices [17]. It can also be used for long-ranged $C_{ij}$ but only under the condition that the number of orbitals $n$ is large. For $n = 1$, extracting the standard NL$\sigma$M from (12) is a highly non-trivial task for the case of a long-ranged matrix $C_{ij}$ (which is a textbook example of the derivation of NL$\sigma$M in the framework of the HS approach).

This example shows once again that the two exact supermatrix representations of the random matrix ensemble which emerge after the SB and HS transformations, are fundamentally different and largely complementary. Thus the new SB approach does not negate or supersede the standard HS one. Rather, it extends the supersymmetry method to an area previously not accessible to it.

**Acknowledgements** K.B.E. and M.R.Z. acknowledge financial support by the Deutsche Forschungsgemeinschaft (SFB/TR 12).

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